Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Thursday, September 15, 2011, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems

If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1 Electron in a Rotor Trap

Consider an electron in a trapping potential of the form

$$V(x, y, z) = \tfrac{1}{2} m_e \Omega_A^2 (R_0 - \rho)^2 + \tfrac{1}{2} m_e \Omega_B^2 \, z^2, \tag{1}$$

where $\Omega_A, \Omega_B, R_0 > 0$ are constants, m_e is the electron mass and $\rho = \sqrt{x^2 + y^2}$.

(a) Find approximate eigenenergies of the system, assuming that the trapping potential is "deep" in radial direction. In that case, $\ell = \sqrt{\hbar/(m_e \Omega_A)} \ll R_0$, and terms proportional to ℓ/R_0 may be neglected.
[*Hint*: use $\rho^{-1/2} R(\rho)$ for the radial part of the wave function.]

(b) Find the leading corrections to the eigenenergies. Are all degeneracies lifted? Why/why not?

(c) The above approximation scheme works well for states close to the ground state. For a given ratio $\ell/R_0 \ll 1$, when do you expect the approximation to break down?

Laplacian in cylindrical coordinates: $\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

2 Interaction between Atom and Electromagnetic Mode

The minimal model of a single atom interacting with photons inside an electromagnetic resonator is

$$H = \frac{\hbar\Omega}{2}\hat{\sigma}^z + (\hbar\omega + \tfrac{1}{2})\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}^\dagger\hat{\sigma}^- + \hat{a}\hat{\sigma}^+). \tag{2}$$

It takes into account two electronic levels of the atom, written in (pseudo-)spin language as $|\uparrow\rangle$ and $|\downarrow\rangle$, and separated by an energy $\hbar\Omega$. Photons of energy $\hbar\omega$ are created and annihilated by the usual harmonic oscillator ladder operators $\hat{a}$ and $\hat{a}^\dagger$.

(a) The model is exactly solvable because of a special symmetry. Explain the meaning of the operator $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{\sigma}^+\hat{\sigma}^-$ and show that it is a conserved quantity.

(b) Consider the basis composed of all states $|\, n, \sigma_z \,\rangle$ with $n = 0, 1, 2, \ldots$ counting photons, and $\sigma_z = \uparrow, \downarrow$ denoting the atom state. Using the conservation law from (a), show: except for the ground state, all eigenstates and corresponding eigenenergies of H can be found by diagonalizing a 2×2 matrix. (Only find the matrix, do not diagonalize yet.)

(c) For the resonant case, i.e. $\Omega = \omega$, find explicit expressions for eigenstates and eigenenergies. Starting in the initial state $|\, n, \uparrow \rangle$, how does the expected photon number evolve as a function of time?

Useful relations & definition of Pauli matrices:

$\hat{a}^\dagger \,|\, n \,\rangle = \sqrt{n+1}\,|\, n+1 \,\rangle$

$\hat{a}\,|\, n \,\rangle = \sqrt{n}\,|\, n-1 \,\rangle$

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}^\pm = \tfrac{1}{2}(\hat{\sigma}^x \pm i\hat{\sigma}^y)$$

3 Kicking a nucleus

(a) A particle is confined by a potential fixed inside the lab frame. If $\psi(\mathbf{r}, t)$ is its wave function (as used by an observer within the lab frame), what is the corresponding wave function used by an observer moving at constant speed $\mathbf{v}$ relative to the lab frame? Or, in other words: how does the wave function transform under a Galilean boost?

(b) Show that the Schrödinger equation for two particles with masses m and M and interacting via a potential of the form $V(\mathbf{r}_1 - \mathbf{r}_2)$, can be separated into two equations describing their relative and center of mass motion.

(c) Use your results from (a) and (b) to solve the following problem. A hydrogen atom in its ground state is initially at rest in the laboratory frame. Due to a collision, the nucleus experiences an abrupt "kick" at time $t = 0$, after which it moves with velocity $\mathbf{v}$ in the lab frame. Find the probability for the atom to remain in its ground state, and discuss the limits of small and large velocity.

Hydrogen wave functions: $\psi_{nlm}(\mathbf{r}) = R_{nl}(r)\, Y_{lm}(\theta, \varphi)$

$$n = 1,\ l = 0,\ m = 0: \qquad R_{nl}(r) = 2a^{-3/2} \exp(-r/a), \qquad Y_{lm}(\theta, \varphi) = 1/\sqrt{4\pi}$$

4 Generating and destroying entanglement

Consider a system composed of two spins, each with $S = \frac{1}{2}$. A state of this system is called *separable* if it can be written in the form $|\,\Psi\,\rangle = |\,\text{state of spin 1}\,\rangle \otimes |\,\text{state of spin 2}\,\rangle$. States that are not separable are called *entangled*. An operation that can transform a separable state into an entangled state is said to *generate entanglement*; an operation that can transform an entangled state into a separable state is said to *destroy entanglement*.

(a) Consider the states $|\,\Psi_a\,\rangle = (|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle)/2$ and $|\,\Psi_b\,\rangle = (|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)/\sqrt{2}$. For each of them, show whether the state is entangled or not.

(b) Assume that the system evolves under the Hamiltonian $H = H_1(t) + H_2(t)$, where $H_j(t)$ acts on spin j. Explain whether or not H can generate entanglement.

(c) Show that the Hamiltonian $H_{12} = \hbar J(\hat{\sigma}_1^+ \hat{\sigma}_2^- + \hat{\sigma}_1^- \hat{\sigma}_2^+)$, acting on the two spins for some finite time Δt, can generate entanglement.

(d) Using the states from (a), give an example for a measurement that destroys entanglement, and an example for a measurement that generates entanglement. For each, answer: Is the process deterministic? If not, what is the probability to obtain the desired outcome?

Definition of Pauli matrices:

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}^{\pm} = \tfrac{1}{2}(\hat{\sigma}^x \pm i\hat{\sigma}^y)$$