Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, September 21, 2012, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Problem 1

Consider a spin-1 system and let

$$|\,1\,\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\,0\,\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\,-1\,\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

denote the eigenstates of the S_z operator, which form an orthonormal basis.

(a) For the state $|\,\Psi\,\rangle = \frac{1}{\sqrt{2}}(|\,0\,\rangle + i\,|\,-1\,\rangle)$, determine $S_z\,|\,\Psi\,\rangle$ as well as the expectation value $\langle\,\Psi\,|\,S_z\,|\,\Psi\,\rangle$.

(b) Using the above orthonormal basis and the state $|\,\Psi\,\rangle$ from (a), evaluate the density matrix

$$\rho = \alpha|\,0\,\rangle\langle\,0\,| + \beta|\,\Psi\,\rangle\langle\,\Psi\,|.$$

Which condition(s) – necessary and sufficient – must the coefficients α and β satisfy so that ρ is a valid density matrix?

(c) Provided the spin is prepared in the state ρ defined in (b), find the expectation value for the measurement of the observable

$$B = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{pmatrix}.$$

(d) Obtaining the expectation value in (c) requires an ensemble average over a large number of measurements. In each individual measurement, what are the possible measurement results and their corresponding probabilities of occurence when $\alpha = 1/2$?

Problem 2

A time-dependent force

$$\mathbf{F}(t) = F_0 \, \mathbf{e}_x \, e^{-t/\tau} \theta(t)$$

acts on an isotropic, two-dimensional harmonic oscillator.[1] For times $t < 0$ the oscillator is in the ground state ($n_x = n_y = 0$).

(a) Express the Hamiltonian of the system in the form $H = H_0 + V(t)$. Here, H_0 is the time-independent Hamiltonian describing the unperturbed harmonic oscillator for $t < 0$.

(b) For the case of a weak force, calculate the approximate probability $\mathcal{P}_1$ of finding the oscillator in either of the first excited states (eigenstates of H_0) at time $t > 0$.

(c) Your approximation in (b) requires that the force be small in some appropriate sense. Discuss the concrete condition(s) under which your approximation is expected to be good.

(d) Is there a nonzero probability to find the oscillator in a higher excited state $n_x > 1$ in the limit $t \to \infty$? Justify your answer with a calculation.

(e) Discuss the limit $\tau \to \infty$ and relate your result for $\mathcal{P}_1$ from (b) to the approximation for sudden changes of the Hamiltonian.

Useful relations for part (e):
Normalized eigenfunctions of the 1-d harmonic oscillator (mass m, angular frequency ω):

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \ell_{\text{osc}} \sqrt{\pi}}} H_n(x/\ell_{\text{osc}}) e^{-\frac{1}{2}(x/\ell_{\text{osc}})^2}, \qquad \text{where } \ell_{\text{osc}} = \sqrt{\hbar/(m\omega)}.$$

Definite integral of Gaussian and Hermite polynomial with shifted center point:

$$\int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} H_n(\xi + a) = 2^n \sqrt{\pi} (-a)^n.$$

[1]Here, $\mathbf{e}_x$ is the unit vector in x direction and $\theta(t)$ denotes the Heaviside step function, defined piece-wise as $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t > 0$.

Problem 3

Positronium is the hydrogen-like bound state of an electron and a positron (anti-particle of the electron, same mass as the electron but opposite electric charge).

(a) Based on the analogy with the hydrogen atom, describe the positronium energy spectrum and the corresponding quantum numbers. In this part, ignore fine splitting, hyperfine splitting and other relativistic corrections.

(b) Next, consider effects due to the electron and positron spin. Give a heuristic derivation of the Hamiltonian term describing spin-orbit interaction. (You may disregard the issue of Thomas precession.) How does the positronium fine structure compare to the fine structure of the hydrogen atom?

(c) For positronium states with vanishing orbital angular momentum, fine splitting does not occur. However, hyperfine splitting due to spin-spin interaction and coupling to an external magnetic field $\mathbf{B} = B_0\,\mathbf{e}_z$ in z may cause level shifts. Their contributions to the Hamiltonian are given by

$$H_{\text{spin}} = -\Lambda \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 - B_0(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \cdot \mathbf{e}_z,$$

where the subscripts 1 and 2 refer to the electron and positron, respectively; $\Lambda > 0$ is a constant. In the absence of an external magnetic field, what are the energy shifts from H_{spin}?

(d) What are the shifts from H_{spin} if the magnetic field is nonzero?

You may use $m_p/m_e \approx 2000$ for the ratio between proton (not positron!) and electron mass.

Problem 4

Consider the scattering potential

$$V(\mathbf{r}) = \lambda\, r^{-1} e^{-r/r_0},$$

where λ and r_0 are both real and positive constants, and $r = |\mathbf{r}|$.

(a) A particle of mass m with sufficiently high energy is weakly scattered by the potential $V(\mathbf{r})$. Give an approximate expression for the differential cross section $d\sigma/d\Omega(\theta)$ for this case.

(b) Consider two particles with interaction potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ as given above. How can the solution of this two-particle problem be reduced to a problem similar to (a)?

(c) Consider two indistinguishable spin-1/2 fermions, prepared in an $S = 1$ state, and subject to the interaction potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ from above. Before the collision, you may consider the two particles to be in plane-wave states with momenta $\pm p\,\mathbf{e}_z$. Do you expect the differential cross section to be the same as in (a)? Explain and verify your answer by explicit calculation of the cross section $d\sigma/d\Omega$ at $\theta = \pi/2$ in the reference frame with zero total momentum. (Use the usual scattering ansatz.)

(d) How does $d\sigma/d\Omega|_{\theta=\pi/2}$ change if the two fermions from (c) are instead prepared in the $S = 0$ state?