Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Tuesday, September 15, 2015, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. A hydrogen atom is placed in an external electric field $\mathcal{E}$ along the z axis. Suppose that the atom is prepared in a state $|\psi\rangle$ with the wavefunction,

$$\psi(\mathbf{r}) = (\text{const})(1 + \alpha z)e^{-r/a_0},$$

where α is a constant, $r = (x^2 + y^2 + z^2)^{1/2}$, and a_0 is the Bohr radius, given by

$$a_0 = \begin{cases} 4\pi\epsilon_0\hbar^2/me^2, & \text{(SI)}, \\ \hbar^2/me^2, & \text{(Gaussian)}. \end{cases}$$

Note 1: The calculations in this problem can be greatly simplified by

(i) using symmetry, and not doing the same integral over and over again,

(ii) noting that when $\alpha = 0$, $\psi(\mathbf{r})$ gives the ground state of H in zero electric field.

(a) Normalize $\psi(\mathbf{r})$.

(b) In the state $|\psi\rangle$ find the expectation value of the zero-field hydrogen Hamiltonian.

(c) In the state $|\psi\rangle$ find the expectation value of the potential energy perturbation due to the electric field.

(d) We now wish to treat $\psi(\mathbf{r})$ as a variational wavefunction, with α as a variational parameter. Find the best possible estimate to the energy of the atom in its ground state in the presence of the field $\mathcal{E}$, *assuming that $\mathcal{E}$ is small* (see note 2).

[**Note 2:** If $\mathcal{E}$ is not small, the ground state energy is a complicated function of $\mathcal{E}$. Because we are only interested in *small* $\mathcal{E}$, you should find your variational estimate for the energy as a series in $\mathcal{E}$, going up to order $\mathcal{E}^2$ only.]

2. (a) Consider a particle of mass m in the 1d potential well drawn in Fig. 1. There is an infinite wall at $x = 0$, i.e., $V(x) = \infty$ for $x \leq 0$.

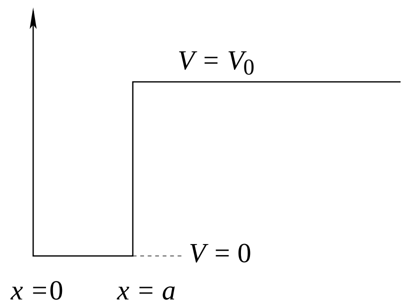

Fig. 1

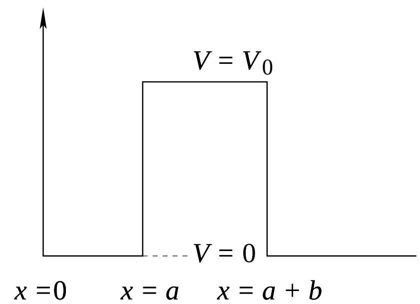

Fig. 2

Suppose, first, that $V_0 = \infty$. What is the energy of the ground state? You may just state the answer from memory if you know it.

(b) Now suppose $\hbar^2/2ma^2 \ll V_0 < \infty$. Find the wavefunction of the ground state, and call it $\psi_1(x)$. Do not try to normalize the wavefunction.

(c) Continuing part (b), find the energy of the ground state. Your answer should be an explicit expression in terms of constants and parameters of the problem, and you must keep the first nonzero correction due to the small parameter $\hbar^2/2ma^2V_0$. (In other words, your answer should not be the same as that for part (a).)

(d) Now consider the potential in in Fig. 2, and denote the eigenfunction for the same energy as in part (c) by $\psi_2(x)$. Clearly,

$$\psi_2(x) = \psi_1(x) \quad \text{if} \quad x < a + b,$$
$$\psi_2(x) \neq \psi_1(x) \quad \text{if} \quad x > a + b.$$

Find $\psi_2(x)$ everywhere by solving Schrodinger's equation for $x > a+b$ and imposing physically appropriate conditions on the wavefunction at $x = a+b$. Again, ignore normalization.

(e) From $\psi_2(x)$, find the ratio

$$\tau = P_{\rm w}/j_{\rm out},$$

where $P_{\rm w}$ is the probability of finding the particle in the well from $x = 0$ to $x = a$, and $j_{\rm out}$ is the outbound part (i.e. the part moving to the right) of the probability current. Simplify your answer as much as possible.

(f) Give a **brief** explanation in words why τ is the lifetime of the metastable ground state in the potential of Fig. 2.

3. A one dimensional harmonic oscillator of mass m and frequency ω is described by the time-dependent Hamiltonian

$$\mathcal{H}(t) = \hbar\omega(a^\dagger a + \tfrac{1}{2}) - f\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)g(t),$$

where f is a constant, and

$$g(t) = \begin{cases} 0 & t \le 0, \\ 1 - e^{-t/\tau} & t > 0, \end{cases}$$

where τ is a positive constant. The oscillator is prepared in its ground state for $t < 0$.

Potentially useful information:

(i) Solution to a first order differential equation:

$$\text{If} \quad \frac{dy}{dt} = \alpha y + h(t), \quad \text{then} \quad y(t) = e^{\alpha t}y(0) + \int_0^t e^{\alpha(t-t')}h(t')\,dt'.$$

(ii) Position and momentum operators are given by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger).$$

(a) Solve the Heisenberg equations of motion for the Heisenberg picture operators $a(t)$ and $a^\dagger(t)$ for $t > 0$.

(b) Remembering that the oscillator is in its ground state for $t \le 0$, use your answers from part (a) to find the mean values of the position and momentum, $\langle x \rangle$ and $\langle p \rangle$, for $t > 0$.

(c) Suppose $\tau \gg \omega^{-1}$. Find $\langle x \rangle$ and $\langle p \rangle$ as $\omega\tau \to \infty$ keeping the ratio t/τ fixed. What conclusions can you draw about the time evolution of the state of the particle, and what general principle of quantum mechanics do your answers illustrate?

(d) Now suppose $\tau \ll \omega^{-1}$. Find $\langle x \rangle$ and $\langle p \rangle$ as functions of t in the limit $\omega\tau \to 0$. What conclusions can you draw about the time evolution of the state of the particle, and what general principle of quantum mechanics do your answers illustrate?

4. A particle of mass m and wavevector k (energy $\hbar^2 k^2/2m$) scatters from the spherically symmetric potential

$$V(r) = \frac{g}{r} e^{-\mu r},$$

where g and μ are positive constants. Using the first Born approximation find

(a) the scattering amplitude $f(\theta, \varphi)$, where (θ, φ) are the spherical polar coordinates of the scattering direction with respect to the direction of the incident particle,

(b) the differential and total scattering cross section.

(c) What is the condition for the Born approximation to be valid? Interpret this condition **briefly** in words.

(d) At sufficiently low energies, scattering in all angular momentum channels except the s-wave ($\ell = 0$) becomes negligible. Find the condition on the energy or wavevector for this to be so. Interpret this condition **briefly** in words.