Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Tuesday, September 17, 2013, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. Two identical spin-1/2 particles interacting via a central potential scatter off each other while in a state of total spin 1.

   (a) Write a general form for the wave function of this system.

   (b) Determine the scattering angle θ (if any) for which the differential cross-section $d\sigma/d\Omega$ vanishes.

   (c) If the interaction can be described by V(r) = $g\frac{e^{-r/a}}{r}$ where g is a real constant, compute the angular dependence of the differential cross-section $d\sigma/d\Omega$ in the Born approximation for identical fermions, total spin 1.

2. When electrons are injected into liquid He at $T \approx 0$, spherical bubbles form around them. The surface tension (equivalently, the energy per unit surface area) of liquid He is σ.

(a) Find the ground state and first excited state energies of a particle trapped in an infinitely deep spherical well of radius R.

(b) Assume that the total energy is a sum of the surface energy and electron energy only (i.e. neglect any other contributions). Treating R as a classical (continuous) variable, determine the radius of a bubble containing one electron at $T=0$.

(c) Find the radius of a bubble containing 4 electrons, neglecting interactions between electrons.

3. An electron is moving in one dimension subject to a periodic boundary condition (period L).

(a) Consider the electron to be a free particle (apart from the boundary condition). Find the stationary wave functions of the system. What is the degeneracy of these states?

(b) Now add a perturbation $V(x)=\varepsilon \cos(\kappa x)$ where ε is small and κ is a large multiple of $2\pi/L$. Determine the energy levels and stationary states to first order in ε for an electron momentum of $\hbar\kappa/2$.

(c) Calculate the correction to the energy of order ε^2.

(d) Determine the energy levels, to first order in small quantities, for electron momentum $\hbar\kappa/2 \pm \Delta$ where Δ is non-zero but very small.

4. The Hamiltonian for the one-dimensional harmonic oscillator can be written

$$H = \hbar\omega(\mathbf{a}^\dagger\mathbf{a}+½) \qquad \text{where } \mathbf{a} \equiv (m\omega/2\hbar)^{½}\,(\mathbf{x}+i\mathbf{p}/m\omega)$$

(a) Find the equation of motion for $\mathbf{a}(t)$ in the Heisenberg picture.

(b) Solve the equation of motion to find the explicit form of $\mathbf{a}(t)$ in terms of $\mathbf{a}(0)$.

(c) Find $\mathbf{x}(t)$ in terms of $x(0)$.

(d) Suppose the oscillator is in its ground state for $t < 0$. A constant force f is applied from $t=0$ to $t=\tau$. To leading order in f, find the probability that the oscillator will be in an excited state, indicating which transitions can occur.

(e) What is the criterion for your answer for (d) to be accurate, in the two limits $\omega\tau >> 1$ and $\omega\tau << 1$?