Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Tuesday, June 9, 2015, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with “1 out of 2”, “2 out of 2” and so on.

Problem 1

A particle of mass M and charge q sits in an infinite 1-D square well with full-width a. A constant electric field E is applied to the system (see figure).

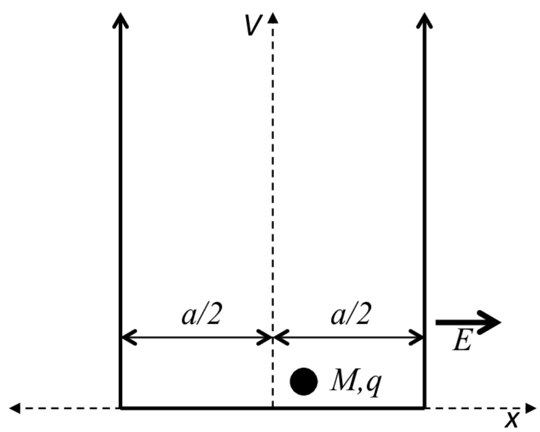
You may leave your answers to the following as infinite sums, but evaluate all integrals. Let $x = 0$ and $V = 0$ at the center of the well.

(a) Give the energy eigenvalues and corresponding wave functions for this system when the electric field is off ($E = 0$).

(b) Use time-independent perturbation theory to find the ground state wave function of the particle when $E \neq 0$, to first order in E.

(c) To first order in E, what is the probability that a measurement of the position of the particle in its ground state will show the particle to be in the right-hand half of the well ($x > 0$)?

(d) Under what conditions will the perturbative approach above be a good approximation?



Possibly useful formulae:

$$\int \mathrm{d}x\, x \sin kx = -\tfrac{x}{k}\cos kx + \tfrac{1}{k^2}\sin kx$$

$$\int \mathrm{d}x\, x \cos kx = \tfrac{x}{k}\sin kx + \tfrac{1}{k^2}\cos kx$$

$$\sin ax \, \cos bx = \tfrac{1}{2}\big(\sin(a-b)x + \sin(a+b)x\big)$$

$$\cos ax \, \cos bx = \tfrac{1}{2}\big(\cos(a-b)x + \cos(a+b)x\big)$$

$$\sin ax \, \sin bx = \tfrac{1}{2}\big(\cos(a-b)x - \cos(a+b)x\big)$$

Problem 2

Consider a particle with mass M and charge q in an isotropic, 3-D harmonic oscillator, given by the potential $U = \frac{1}{2}k\left(X^2 + Y^2 + Z^2\right)$

(a) Show that the Hamiltonian for this system $H = (P^2/2M) + U$ is equivalent to $H = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}\right)$. Remember

$$a_x = \sqrt{\frac{M\omega}{2\hbar}}\left(X + \frac{i}{M\omega}P_x\right)$$

(b) Using the $|n_x, n_y, n_z\rangle$ basis of energy eigenstates, find the energy, degeneracy, and parity of the first three energy levels.

(c) The Hamiltonian is spherically symmetric, so there there must also exist a $|n_r, l, m\rangle$ basis for energy eigenstates, where l and m are the usual $\mathbf{L}^2$ and L_z quantum numbers. Based on the degeneracies and parities found in part (b), what l values are possible at each of the first three energy levels?

(d) Suppose you have prepared the state $|n_x, n_y, n_z\rangle = |1, 0, 0\rangle$. You turn on a magnetic field $B\hat{k}$ for a period of time τ, then turn the field off. What are the expectation values for n_x, n_y, and n_z after the field has been turned off? Give the exact answer, do not use perturbation theory.

Hint: Express $H_B = -\frac{Bq}{2M}L_z$ in the $|n_x, n_y, n_z\rangle$ basis by writing L_z in terms of the $a, a^\dagger$ operators.

Possibly useful formulae:

$$L_z = XP_y - YP_x$$

$$a_x\left|n_x\right\rangle = \sqrt{n_x}\left|n_x - 1\right\rangle, \quad a_x^\dagger\left|n_x\right\rangle = \sqrt{n_x + 1}\left|n_x + 1\right\rangle$$

Problem 3

Alice is sending Bob a string of random qubits — *i.e.,* a sequence of spin-1/2 particles prepared in random spin states known only to Alice. Bob measures the spin-projection of these particles on an axis of his choosing, also random. (This happens to the first step in a quantum cryptography technique.)

Bob and Alice are worried that Eve may be listening in, making her own spin measurement on each qubit and then sending the qubit on to Bob. How can Alice and Bob determine whether Eve is listening in? Answer the question in the following steps:

(a) For each qubit, we can choose the coordinate frame where Bob's measurement defines the z-axis and the state prepared by Alice is given by

$$|\psi\rangle = \cos\frac{\theta}{2}\,|{\uparrow}\rangle + \sin\frac{\theta}{2}\,|{\downarrow}\rangle\,,$$

where $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ are the eigenstates of S_z. Assuming that Eve is *not* listening in, what are the probabilities that Bob measures spins of $+\hbar/2$ and $-\hbar/2$, respectively?

(b) Now suppose Eve *is* listening in, making her own measurement along an axis defined by polar coordinates θ' and ϕ', then passing the qubit on to Bob. What is the probability that Bob measures a spin of $+\hbar/2$ in this case? Give your answer in terms of θ, θ', and ϕ'.
Hint: The eigenstates of $S_{\theta,\phi}$ (*i.e.* the spin-projection into the axis defined by polar coordinates θ and ϕ) are given by

$$\begin{aligned} |{\uparrow}_{\theta,\phi}\rangle &= \cos\frac{\theta}{2}\,|{\uparrow}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|{\downarrow}\rangle \\ |{\downarrow}_{\theta,\phi}\rangle &= \sin\frac{\theta}{2}\,|{\uparrow}\rangle - e^{i\phi}\cos\frac{\theta}{2}\,|{\downarrow}\rangle \end{aligned}$$

(c) Since Bob and Alice's reference frame was chosen at random for each qubit, Eve's θ' and ϕ' in can be taken to be random as well — that is, ϕ' has a uniform distribution in the interval $[0, 2\pi]$, and $\cos\theta'$ has a uniform distribution in the interval $[-1, 1]$. Integrate over these θ' and ϕ' distributions to find the probability that Bob measures a spin of $+\hbar/2$ in terms of only θ. Compare your answer to the case in part (a).

Problem 4

Consider two **identical** spin-1/2 particles of mass m scattering off each other at low energy in the center-of-mass frame. The potential between the two particles is given by the Yukawa coupling, $V(r) = \frac{g}{r}e^{-\mu_0 r}$. The spins of the particles do not interact in this problem. For *distinguishable* particles, the scattering amplitude for this potential in the Born approximation is given by

$$f(\theta) = -\frac{mg}{\hbar^2}\left[\mu_0^2 + 2k^2(1-\cos\theta)\right]^{-1},$$

where the incoming particles have momenta $\pm\hbar k$.

(a) Under what condition is the Born approximation a good approximation? Give a condition that is independent of beam energy (*i.e.* your expression should not include k).

(b) Under what condition will the low-l partial waves dominate the scattering amplitude?

For the remainder of the problem, assume both of the above conditions hold. Give scattering amplitudes to order k^2 and cross sections to order k^4.

(c) Find the $l = 0$ and $l = 1$ components of the scattering amplitude.

(d) Suppose the beams are polarized such that $S_{1z} = S_{2z} = +\hbar/2$. What is the differential cross section for detecting *either* particle at angle θ, where $\theta = 0$ is the initial direction of particle 1?

(e) Suppose instead that the beams are polarized such that $S_{1z} = +\hbar/2$ and $S_{2z} = -\hbar/2$. What is the differential cross section for detecting *either* particle at angle θ in this case?

(f) Suppose the beams are *unpolarized*. What is the differential cross section for detectiont *either* particle at angle θ in this case?

Possibly useful formulae:

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1) \qquad P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$\int_{-1}^{1} d[\cos\theta] P_m(\cos\theta) P_n(\cos\theta) = \frac{2}{2n+1}\delta_{mn}$$