Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Thursday, June 7, 2012, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Problem 1

Consider an infinite chain of quantum dots in which one electron can hop between nearest neighbor quantum dots, described by the Hamiltonian

$$H = -V \sum_{n=-\infty}^{\infty} \Big[|\, n+1 \,\rangle\langle\, n \,| + |\, n \,\rangle\langle\, n+1 \,| \Big].$$

Here, $V > 0$ is the hopping matrix element and $|\, n \,\rangle$ denotes the state in which the electron occupies the n-th dot ($n = 0, \pm 1, \pm 2, \ldots$). The states are orthonormal: $\langle\, n \,|\, m \,\rangle = \delta_{nm}$.

(a) Verify that the eigenstates of H are given by the states

$$|\, k \,\rangle = C \sum_{n=-\infty}^{\infty} e^{ikn} \,|\, n \,\rangle\,,$$

and find the corresponding eigenenergies. What values should k take so that the set $\{|\, k \,\rangle\}$ forms a basis of the Hilbert space? Determine the normalization constant C such that $\int_{-\pi}^{\pi} dk \,|\, k \,\rangle\langle\, k \,| = \mathbb{1}$.

(b) At time $t = 0$, the electron is localized on the quantum dot $n = 0$. Calculate the probability $P_m(t)$ to find the electron on quantum dot $n = m$ at a later time $t > 0$. What is the asymptotic behavior of the probability to find the electron on its initial quantum dot ($n = 0$) at large times?

(c) For a short time interval $\Delta t \ll \hbar/V$, evaluate the probability $P_0(\Delta t)$, keeping only terms up to order $(\Delta t\, V/\hbar)^2$. If an additional measurement of the electron position n is carried out at time $t = \Delta t/2$, what is the probability $\bar{P}_0(\Delta t)$ to find the electron on the $n = 0$ quantum dot at time Δt? Compare $P_0(\Delta t)$ and $\bar{P}_0(\Delta t)$. Qualitatively, what behavior do you expect if position measurements are performed repeatedly with a high repetition rate $\Gamma \gg V/\hbar$?

The following properties of the Bessel functions $\mathrm{J}_n(z)$ of the first kind are useful to recall:

$$\mathrm{J}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} dx\, e^{iz \sin x - inx},$$

$$\mathrm{J}_{-n}(z) = (-1)^n\, \mathrm{J}_n(z),$$

$$\mathrm{J}_n(z) \simeq \sqrt{\tfrac{2}{\pi z}} \cos(z - \tfrac{1}{2} n\pi - \tfrac{\pi}{4}) \quad \text{as } z \to \infty \ (n \text{ fixed})$$

$$\mathrm{J}_n(z) \simeq \tfrac{1}{n!} \left(\tfrac{z}{2}\right)^n \left[1 - \tfrac{1}{n+1} \left(\tfrac{z}{2}\right)^2\right] \quad \text{as } z \to 0 \ (n \geq 0 \text{ fixed})$$

Problem 2

(a) Consider a system of two spins, each with spin $\frac{1}{2}$, with total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Can the eigenvalues of $\mathbf{S}^2$, S_{1z} and S_{2z} serve as a set of quantum numbers labeling the eigenstates, i.e. $|\, S, S_{1z}, S_{2z}\,\rangle$? Prove your answer.

(b) Three distinguishable particles, each with spin $\frac{1}{2}$ and identical magnetic moment $g\mu_B$, are placed in a uniform magnetic field $\mathbf{B}$. Find the resulting eigenenergies and their degeneracies. How many eigenstates have total spin $S = \frac{1}{2}$?

(c) The three particles from (b) occupy the second lowest energy eigenstate with $S = \frac{3}{2}$. What is the probability to find particle 1 in the $s_{z1} = +\frac{1}{2}\hbar$ state?

(d) The three particles occupy the second lowest energy eigenstate, but are now identical, and have a joint orbital wavefunction antisymmetric under particle exchange. What are the eigenenergies and degeneracies of the allowed spin states?

Problem 3

(a) Consider a time-independent Hamiltonian H and an operator K which is time-independent in the Schrödinger picture. Working in the Heisenberg picture,[1] show that the expectation value of dK_H/dt vanishes for every stationary eigenstate $|\psi\rangle$ of H.

(b) For a particle of mass m, define the operator as $K = \mathbf{r} \cdot \mathbf{p}$, where $\mathbf{r}$ and $\mathbf{p}$ denote particle position and momentum, respectively. Assume the form

$$H = \frac{\mathbf{p}^2}{2m} + U(\mathbf{r}) = T + U(\mathbf{r})$$

for the Hamiltonian and show: the expectation values $\langle T\rangle_\psi$ and $\langle \mathbf{r} \cdot \nabla U\rangle_\psi$ are proportional to each other for all eigenstates $|\psi\rangle$ of H. Determine the proportionality constant c in the resulting relation

$$c\langle T\rangle_\psi = \langle \mathbf{r} \cdot \nabla U\rangle_\psi. \tag{1}$$

(This relation is called the virial theorem.)

(c) Simplify equation (1) for the special case of bound states in a central potential

$$U(\mathbf{r}) = A|\mathbf{r}|^n \quad \text{with } n \neq 0.$$

Obtain explicit expressions for the expectation values of kinetic and potential energies with respect to eigenstates of the 3d isotropic harmonic oscillator and of the hydrogen atom.

[1] Recall: in the Heisenberg picture, operators carry the time dependence and states are stationary.

Problem 4

Consider a particle (mass m) in three-dimensional space subject to the attractive potential

$$V(\mathbf{r}) = -U_0\Theta(R - |\mathbf{r}|)$$

with potential depth $U_0 > 0$ and range $R > 0$.[2]

(a) Express the Hamiltonian in terms of radial and angular contributions. Use separation of variables to obtain the equation determining the radial function $R(r) = u(r)/r$, and state its boundary conditions.

(b) Find the s-wave ($\ell = 0$) scattering solutions: use

$$u(r) = A\sin(kr + \delta)$$

as your ansatz for $r > R$ and determine the phase δ. Relate A to the amplitude of the wavefunction in the inner region, $r < R$. (You may ignore the question of overall normalization for the scattering states.)

(c) The potential may support s-wave bound states with energies in the range $-U_0 < E < 0$. Obtain the transcendental equation determining their eigenenergies.

(d) Solve the transcendental equation from (c) graphically and discuss the number of s-wave bound states.

In spherical coordinates, the Laplacian is given by

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

[2] $\Theta(x)$ denotes the Heaviside step function with $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$.