Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, June 12, 2013, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. We want to calculate perturbations to the ground state energy of a Hydrogen-like atom perturbed by an external static and uniform electric field, $\vec{E} = E\hat{z}$. Let H_0, $|0\rangle$ and $|k\rangle$ denote the unperturbed Hamiltonian, the ground state, and an excited state respectively. The unperturbed ground state energy is $E_0^{(0)} = -e^2/2a$, where $a = \hbar^2/me^2$ is the Bohr radius. The ground state wave function is

$$|0\rangle = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}.$$

We are interested in the first-order correction, $E_0^{(1)}$ and will use an operator

$$F = -\frac{ma}{\hbar^2}\left(\frac{r}{2} + a\right) z \tag{1}$$

to calculate it.

You are given the following facts:

$$z|0\rangle = (FH_0 - H_0F)|0\rangle \tag{2}$$

$$z_{k0} \equiv \langle k|z|0\rangle = (E_0^{(0)} - E_k^{(0)})\langle k|F|0\rangle. \tag{3}$$

$$\sum_{k\neq 0} \frac{|z_{0k}|^2}{E_0^{(0)} - E_k^{(0)}} = \langle 0|zF|0\rangle - \langle 0|z|0\rangle\langle 0|F|0\rangle. \tag{4}$$

$$\langle 0|zF|0\rangle - \langle 0|z|0\rangle\langle 0|F|0\rangle = -\frac{1}{3}\frac{ma}{\hbar^2}\left(\frac{1}{2}\langle r^3\rangle_0 + a\langle r^2\rangle_0\right). \tag{5}$$

(a) Given the definition Eq. (1), prove Eq. (5).

(b) Using Eqs. (4) and (5), find the second-order correction $E_0^{(2)}$ to $E_0^{(0)}$.

(c) Prove Eq. (4), taking Eqs (1) - (3) as true.

(d) Prove Eq. (3) taking Eq. (1) and (2) as true.

(e) Prove Eq. (2) using only the definitions of F and H_0 and the symmetry properties of the state $|0\rangle$.

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \qquad \vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$

$$\vec{\nabla} r = \hat{r} \qquad \nabla^2(fg) = (\nabla^2 f)g + f(\nabla^2 g) + 2\vec{\nabla} f\cdot\vec{\nabla} g$$

2. Two identical $s = 1$ bosons are trapped in a bound state by a central potential, $V(r)$, where $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between the bosons.

   (a) Write down the ground state wave function, assuming that $L = 0$ is not allowed. What is the degeneracy of the ground state?

   (b) A perturbation is introduced that acts in the z-direction:

   $$\delta V = V_1 \, L_z$$

   where V_1 is a constant and $\vec{L}$ is the angular momentum operator.

   How is the degeneracy of the ground state broken?

   (c) A different perturbation is introduced, in place of the perturbation δV specified in part (b),

   $$\delta V' = -V_2 \, \vec{L} \cdot \vec{S}$$

   with $V_2 > 0$. Find the change in the energy of the ground state. What configuration of spins corresponds to the lowest energy state?

   (d) Now consider both perturbations δV and $\delta V'$ together, and assume $\delta V' \gg \delta V$. Focus on the modified ground state created by perturbation $\delta V'$. The perturbation δV splits these states. What is the pattern of the splitting, and how large is the energy difference between two adjacent states?

   Draw and label an energy level diagram.

3. The free particle Hamiltonian is $H = p^2/2M = -(\hbar^2/2M)\,\nabla^2$. Let a particle of mass M and charge Q be confined to a ring of radius R, with

$$\nabla^2 = \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}.$$

The appropriate normalization condition for a wave function $\psi(\phi)$ is

$$\int_{\text{ring}} |\psi(\phi)|^2 \, d\ell = 1$$

where $0 \le \ell \le 2\pi R$.

(a) If E is the eigenvalue of the Hamiltonian, so that $H\,\psi_E = E\,\psi_E$, what are the explicit forms of the functions $\psi_E(\phi)$?

These functions must be single-valued. What constraint does this place on the allowed eigenvalues E_n, and what is the degeneracy of each state? (*Hint:* n will be an integer.)

Express the time-dependent functions as waves traveling clockwise $\Psi_{\text{CW}}(\phi, t)$ and counterclockwise $\Psi_{\text{CCW}}(\phi, t)$ around the ring.

(b) The parity operator ($\mathbf{P} : \phi \to -\phi$) commutes with the Hamiltonian. Find linear combinations of your functions from part (a) that are also eigenfunctions of $\mathbf{P}$, and label them $\Psi_{\pm}(\phi, t)$.

Explain how this set of eigenfunctions can be characterized by their values at $\phi = 0$.

(c) The angular momentum operator has the representation $L_z = (\hbar/i)\,\partial/\partial\phi$. Which of your (time-independent) states are eigenfunctions of L_z, and what are the eigenvalues?

(d) Turn on a constant magnetic field that produces a flux Φ_B through the ring. The complete Hamiltonian is now

$$H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{Qv}{c}A,$$

where $A = \Phi_B/2\pi R$ is the vector potential, and v is the velocity operator.

What are the modified energy levels for this Hamiltonian?

4. Consider a two-level system with ground state $|0\rangle$ and excited state $|1\rangle$.

The Hamiltonian can be written

$$H_0 = \begin{pmatrix} \langle 0|H_0|0\rangle & \langle 0|H_0|1\rangle \\ \langle 1|H_0|0\rangle & \langle 1|H_0|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$$

where $\epsilon > 0$.

At $t = 0$, an interaction $b(t)$ is turned on, where

$$b(t) = \begin{pmatrix} \langle 0|b(t)|0\rangle & \langle 0|b(t)|1\rangle \\ \langle 1|b(t)|0\rangle & \langle 1|b(t)|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_2(t) \\ \epsilon_1(t) & 0 \end{pmatrix}$$

so the Hamiltonian is $H = H_0 + b(t)$ for $t > 0$.

Initially, the system is in the ground state $|0\rangle$.

(a) Find *exact* differential equations for the time evolution for $t > 0$, i.e., equations for $\psi_0(t) = \langle 0|\psi(t)\rangle$ and for $\psi_1(t) = \langle 1|\psi(t)\rangle$.

(b) Suppose $\epsilon_1(t) = (iv/2)\sin(\omega t)$, with v real. Assuming that probability is conserved for $t > 0$, determine $\epsilon_2(t)$.

(c) Assuming that v is sufficiently small, and working in the *interaction picture*, find the probability $P_1(t)$ of observing the excited state, for $t > 0$, to lowest nonzero order in v.

(d) At what frequency ω is the initial increase of $P_1(t)$ the largest?