Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Tuesday, June 10, 2014, 9 am - 1 pm

Quantum Mechanics

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. Particle in a one-dimensional potential

A particle of mass m is in a potential in one dimension x:

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ -2V_0 & a < x < 2a \\ \infty & x > 2a \end{cases}$$

where V_0 and a are positive constants.

Note: The particle is in an eigenstate with energy $E = -V_0$ for all parts of this problem.

(a) Write down the general solutions of the Schrodinger equation in the three regions $x < 0$, $0 < x < a$, and $a < x < 2a$, without considering boundary conditions. Express your results in terms of $k = \sqrt{2mV_0/\hbar^2}$.

(b) Clearly state and apply the boundary conditions, to obtain four equations determining the nonzero amplitudes of the solutions found in (a).

(c) Solve the equations of (b) to obtain a transcendental equation expressing the condition (which depends on m, a and V_0) that there be an eigenstate at $E = -V_0$.
[hint: the form of the equation will be $\tan(ka) = f(ka)$]

(d) Use a graph to indicate how to find the *smallest* value of V_0 (equivalently the smallest value of k) for which there is an eigenstate with $E = -V_0$.

(e) Using your graphical solution, roughly estimate the smallest allowable value of V_0 and sketch the wavefunction for this case.

2. Two-state system

A two-state system has Hamiltonian:

$$\hat{H} = \Omega \cos(\omega t)\Big[\,|+\rangle\langle+| \quad - \quad |-\rangle\langle-|\,\Big]$$

where $|+\rangle$ and $|-\rangle$ are orthonormal.

Consider an operator $\hat{D}$ whose action on $|+\rangle$ and $|-\rangle$ is:

$$\hat{D}\,|+\rangle = d\,|-\rangle$$

$$\hat{D}\,|-\rangle = d\,|+\rangle$$

where d is real and positive.

(a) Write the time-dependent Schrodinger equation for

$$|\psi(t)\rangle = \psi_+(t)\,|+\rangle + \psi_-(t)\,|-\rangle$$

(b) Solve (a) to obtain $|\psi(t)\rangle$, given that at $t = 0$ the system is in the eigenstate of $\hat{D}$ with eigenvalue d.

(c) Given the solution (b), suppose a measurement of $\hat{D}$ is made at time $t > 0$. What is the probability $P(t)$ that the eigenvalue *not equal* to d is measured?

(d) If Ω is below a certain limit Ω_0, $P(t)$ can never be 100%. Find Ω_0.

3. Particle with fixed total orbital angular momentum

A spinless particle has angular momentum quantum number $\ell = 2$, and has Hamiltonian

$$\hat{H} = \frac{3\epsilon}{2\hbar}\hat{L}_z - \frac{\epsilon}{\hbar^2}\left(\hat{L}_x^2 + \hat{L}_y^2\right)$$

where $\epsilon > 0$ is a real constant, and $\hat{L}_i$ are the components of the angular momentum operator.

(a) Find the energy spectrum of $\hat{H}$.

(b) Suppose the wavefunction of this particle is

$$\psi(\theta, \phi) = A\left(\cos\theta + \sin\theta\cos\phi\right)\sin\theta\sin\phi$$

where θ and ϕ are the usual spherical angular coordinates, and where A is a normalization constant. If energy is measured, what energies have *nonzero probability* of being measured?

(c) What is the *average* energy that would be obtained from many energy measurements of particles initially prepared in the state of (b)?

(d) Now suppose one of these particles is initially (at time $t < 0$) in the ground state of $\hat{H}$. A magnetic field pointing in the x-direction is applied, with time dependence magnitude $B_0 e^{-t/\tau}$, for $t > 0$ (the field is zero for $t \leq 0$). Write the perturbation Hamiltonian, given that the magnetic moment of the particle has components $\hat{\mu}_i = \mu_0 \hat{L}_i/\hbar$.

(e) For (d), find the transition probabilities to possible excited states for $t >> \tau$ and to first order in the magnetic field strength. Be sure you indicate what final states are possible at first order in B.

4. Partial wave description of hard-sphere scattering

Consider a nonrelativistic quantum particle of mass m scattering from an immobile hard-sphere potential in three dimensions:

$$V(x) = \begin{cases} 0 & r > a \\ \infty & r < a \end{cases}$$

where r is the radial coordinate.

Recall that a scattering wavefunction as $r \to \infty$ is a linear combination of the incident plane wave and outgoing scattered waves:

$$\psi(x) \to e^{ikz} + \frac{f(\theta, \phi)}{r^n} e^{ikr}$$

with scattering cross-section

$$d\sigma/d\Omega = |f(\theta, \phi)|^2$$

(a) Explain what the power n should be in this scattering wavefunction in the $r \to \infty$ limit.

(b) Find the wavefunction $\psi(x)$ and $f(\theta, \phi)$ by using an expansion in partial waves.

(c) Consider the low-energy limit of (b) (small incident particle energy $E = p^2/(2m)$). Show that only one angular momentum state dominates the scattering in this limit (stating clearly which angular momentum state that is).
What is the condition on incident particle energy for one to be in this limit?

(d) Compute the low-energy limit of $d\sigma/d\Omega$.

(e) Compute the low-energy limit of the total cross-section and compare it to the classical geometrical cross-section presented by the hard sphere potential. Comment on any difference between them.

TABLE 4.3: The first few spherical harmonics, $Y_l^m(\theta, \phi)$.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$	$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1) e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$	$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$

$$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} \, e^{im\phi} \, P_l^m(\cos\theta),$$

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta).$$

$$L_\pm \equiv L_x \pm i L_y.$$

$$L_+|l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\; |l, m+1\rangle$$

$$L_-|l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\; |l, m-1\rangle$$

TABLE 11.1: Spherical Hankel functions, $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$.

$h_0^{(1)} = -i\frac{e^{ix}}{x}$	$h_0^{(2)} = i\frac{e^{-ix}}{x}$
$h_1^{(1)} = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix}$	$h_1^{(2)} = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}$
$h_2^{(1)} = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix}$	$h_2^{(2)} = \left(\frac{3i}{x^3} - \frac{3}{x^2} - \frac{i}{x}\right)e^{-ix}$

$$\left.\begin{aligned} h_l^{(1)} &\to \frac{1}{x}(-i)^{l+1} e^{ix} \\ h_l^{(2)} &\to \frac{1}{x}(i)^{l+1} e^{-ix} \end{aligned}\right\} \text{for } x \gg 1$$

TABLE 4.4: The first few spherical Bessel and Neumann functions, $j_n(x)$ and $n_l(x)$; asymptotic forms for small x.

$j_0 = \frac{\sin x}{x}$	$n_0 = -\frac{\cos x}{x}$
$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$	$n_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$
$j_2 = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$	$n_2 = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$
$j_l \to \frac{2^l l!}{(2l+1)!} x^l,$	$n_l \to -\frac{(2l)!}{2^l l!} \frac{1}{x^{l+1}}$, for $x << 1$.

$$h_l^{(1)}(x) \equiv j_l(x) + i n_l(x); \quad h_l^{(2)}(x) \equiv j_l(x) - i n_l(x).$$