Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, June 11, 2010, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. (a) Show that for a bound Keplerian orbit the total energy can be expressed as

$$E = -\frac{GMm}{(r_{\rm max} + r_{\rm min})},$$

where M and m are the masses of the two bodies, G is Newton's constant, and $r_{\rm max}$ and $r_{\rm min}$ are, respectively, the maximum and minimum separations between the two bodies. Throughout, you may assume that $M \gg m$ such that the total mass of the system is approximately equal to M ("the star") and the reduced mass approximately equal to m ("the planet"). Remember that the gravitational potential energy U of a system of two masses M and m separated by a distance r is

$$U(r) = -\frac{GMm}{r}.$$

Now imagine that half of the star's mass instantaneously disappears (you can assume that $M/2$ is still much larger than m). What happens to the orbit of the planet if

(b) its orbit before the disappearing-mass phenomenon was a perfect circle? Sketch the new orbit.

(c) its orbit was an ellipse, and the disappearing-mass phenomenon occurs when the planet was as close to the star as possible ($r = r_{\rm min}$)? Sketch the new orbit.

(d) its orbit was an ellipse, and the disappearing-mass phenomenon occurs when the planet was as far from the star as possible ($r = r_{\rm max}$)? Sketch the new orbit.

2. A particle of mass m and charge q is constrained to move without friction on a vertical circumference of radius R in the presence of a constant downwards gravitational field $\vec{g}$ (see figure). Another charge Q is fixed at the lowest point in the circle.

   (a) What is the Lagrangian of the system? Be sure to clearly define your coordinates.

   (b) Write down the equation(s) of motion.

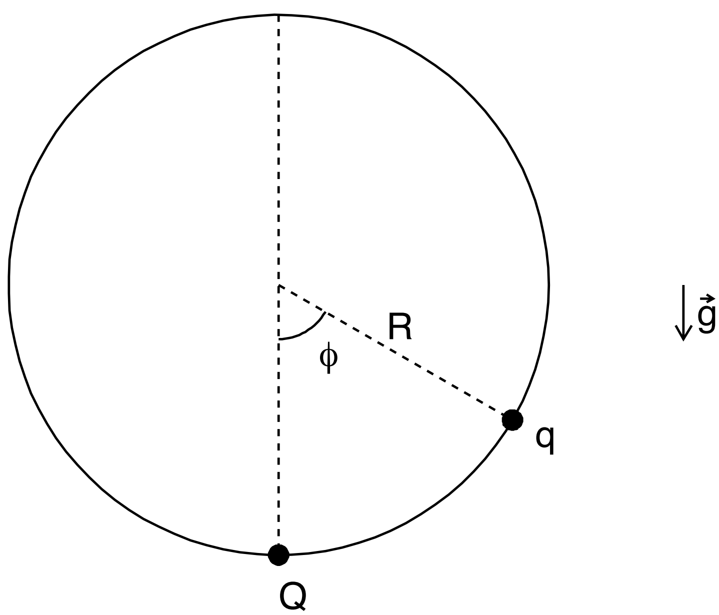
   (c) Determine all the equilibrium configurations of the system and describe the stability of the equilibrium, assuming that the two charges have the same sign.

   (d) Determine the frequency of small oscillations around all stable equilibrium points.

   Remember that the force between two point charges q and Q is, assuming that none of the charges are moving around very fast (and using MKS units),

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\hat{r}}{r^2},$$

   where $\vec{r} = r\hat{r}$ is the relative position vector between the two charges and ϵ_0 is a constant.



3. The Lagrangian for a certain system with two degrees of freedom can be written as

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\alpha\dot{q}_1\dot{q}_2\right) - \frac{\omega_0^2}{2}\left(q_1^2 + q_2^2 + 2\beta q_1 q_2\right),$$

where $\alpha, \beta \geq 0$ are non-negative real numbers and ω_0 is a real number.

(a) Write down the equations of motion and discuss the equilibrium configuration(s) of the system.

(b) For what values of α and β is the equilibrium stable? Ignore for now the possibility that $\alpha = 1$. Compute the normal frequencies of the system along with the normal modes.

(c) For $\alpha = 1/2, \beta = 1$, compute $q_1(t)$ and $q_2(t)$ given that, at $t = 0$, $q_1 = q_2 = \dot{q}_1 = 0$, while $\dot{q}_2 = v_0$.

(d) What happens when $\alpha = 1$ (for any positive β)? Describe the general motion of the system in this case.

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Consider a cubic container of edge L (volume $V = L^3$) containing an ideal gas of N particles, initially at temperature T.

(a) Near the surface of the earth, the gravitational field acting on all the particles of the gas is g (acting in the height direction). Find the density of the gas as a function of vertical position in the container, $\rho(z)$.

(b) Calculate the entropy of the gas as a function of N, the volume V and g, in the small-g limit $mgL/k_BT << 1$ (find the limiting behavior, don't just set $g = 0$!).

(c) Now, suppose the container is launched into deep space, so that now no gravitational field acts on the gas. Under the condition that the temperature of the container is held fixed during the flight, find the change in entropy relative to (b), again in the limit $mgL/k_BT << 1$. Explain your answer.

(d) Suppose you carry out the same experiment, except that now the container is heavily insulated so no heat can be transferred to or from the gas; it starts on earth at temperature T. Find the final temperature T_f of the gas when it reaches deep space where $g = 0$. Explain your answer.

2. Consider a molecule which can be changed in length by the application of force. The force needed to stretch the molecule to length X is $f(X,T)$ where T is temperature.

(a) Find an expression for the Helmholtz free energy $A(X,T)$ up to a constant, in terms of $f(X,T)$.

(b) Explain how the difference in entropy between two lengths X_0 and X_1 at fixed temperature T, $S(X_1,T) - S(X_0,T)$, may be determined from measurements of $f(X,T)$.

(c) Now consider the specific case of a molecule composed of N units each of which may be in one of two states which have lengths ℓ and $\ell + \Delta$. The energy of a unit in the longer state is ϵ higher in energy than a unit in the shorter state. Find $f(X,T)$ for this specific microscopic model.

3. Consider a nanoscale motor which repeatedly undergoes a chemically driven cycle. Because the motor is so small, fluctuations cause the time duration of successive cycles τ to be randomly and independently distributed according to

$$P(\tau) \propto e^{-r\tau}$$

where r is a constant.
The distribution is defined such that the probability of observing durations between τ and $\tau + d\tau$ is equal to $P(\tau)d\tau$.

(a) Find the average rate of cycling of the motor (number of cycles per unit time).

(b) Suppose you observe N cycles. The total time required for those N cycles to occur, $T_N = \tau_1 + \tau_2 + \tau_3 + \cdots + \tau_N$ will be randomly distributed. Find the standard deviation of the distribution of T_N that you would obtain from a long series of observations of N-cycle runs of the motor.

(c) Find the probability distribution of T_N, $P_N(T_N)$.

(d) Explain why the exponential form of the distribution $P(\tau)$ is likely to be observed for operation of a small (think single-molecule) motor.