Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Friday, September 17, 2010, 9 am - 1 pm

Classical and Statistical Mechanics

This examination has two parts, Classical and Statistical Mechanics. You need to complete two out of three problems on both sections of the examination for a total of four solved problems.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

Classical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Consider a pendulum which consists of a massless stick of length ℓ, at the end of which is a homogeneous disk-shaped weight of radius R and mass M which is able to rotate around a point near its edge. The rotational axis of the disk is parallel to the rotational axis of the pendulum stick (see picture). and the pendulum swings in gravitational field $-g\hat{\mathbf{z}}$.

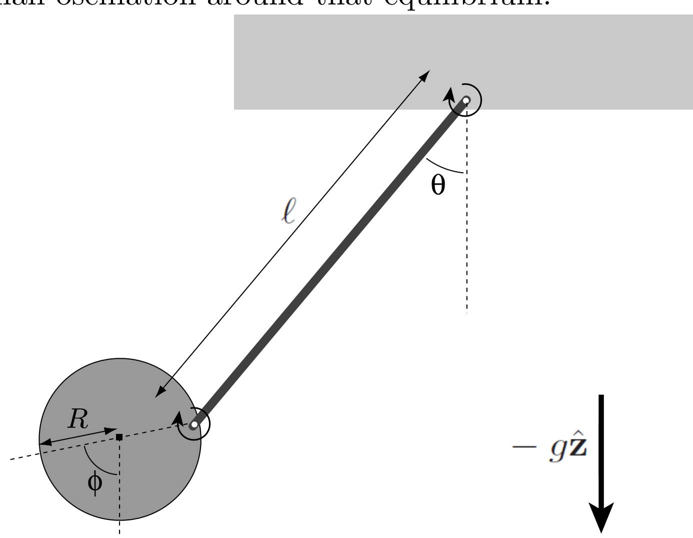
(a) Find the moment of intertia of the disk about its center of mass (I_{CM}), and about its pivot (I_{pivot}).

[parts (b)-(d) may be solved independently of (a)]

(b) Write the Lagrangian of this system in terms of the two angles θ and ϕ (see picture).

(c) Find the equations of motion for θ and ϕ (do not solve).

(d) Show that $\theta = \phi = 0$ is a stable equilibrium point, and find the frequencies for the normal modes of small oscillation around that equilibrium.



2. Consider a "space elevator" consisting of a long rod in geostationary orbit around the earth with its lower end just above the equator, that could be used to take objects into space.

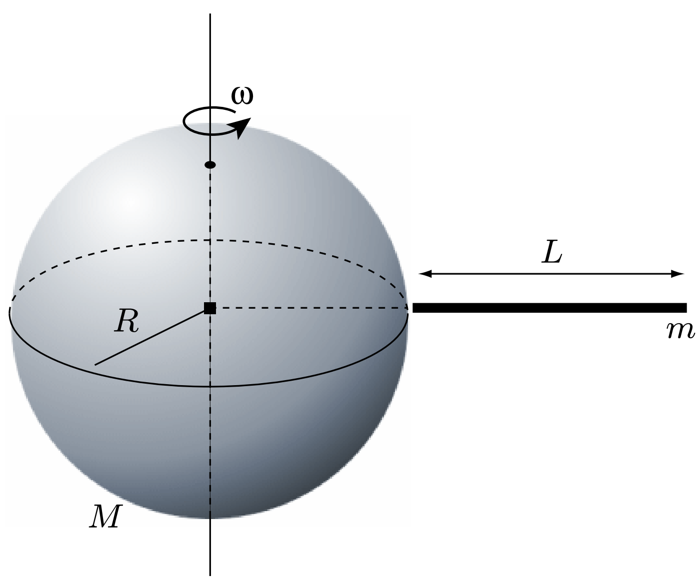
Suppose the rod is of length L and mass m, and the earth is of radius R and mass M. The earth is rotating at constant angular velocity ω. You may neglect the effect of the rod on the motion of the earth.

(a) Find an expression for the tension T in the rod as a function of height z above the earth's surface. (Hint: the tension is the force acting across a cross-section of the rod; positive tension at z is the force that would need to be applied to hold the two pieces of the rod together after cutting it; negative tension at z is the force that would be applied from one piece to the other after cutting it).

(b) Find an expression for the length of the rod L for which it will hang freely above the earth's surface.

(c) Estimate the length of the rod from (b), and also the *maximum* tension along the rod. The earth has radius $R = 6400$ km, and $g = 9.8$ m/s^2 is the gravitational acceleration at the earth's surface.

(d) Explain what happens for a rod either slightly longer or slightly shorter than the length obtained in (b).



3. Consider orbits in the field of an attractive central force field $\mathbf{f}(r) = -kr^n\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is the radial unit vector.

(a) Find the potential energy associated with the force field.

(b) Find the Lagrangian and the equations of motion for orbits confined to a plane including the center of the potential, using polar coordinates r and θ.

(c) Find the relation between orbital period and radius for *circular* orbits centered on the center of the potential.

(d) For what ranges of n are the circular orbits stable or unstable to small perturbations?

Statistical Mechanics - solve 2 out of 3 problems

If you solve all 3 problems, only the first 2 will be graded.

1. Suppose that some tiny particles of mass m are initially dispersed *uniformly* in the still air of a room of height h at number density n_0 at temperature T, where gravitational acceleration is a constant $-g\hat{\mathbf{z}}$.

The particles can be considered to be spheres of radius R, and they move in response to an external force $\mathbf{F}$ at a constant drift velocity $\mathbf{u} = \mathbf{F}/(6\pi\eta R)$, where η is the viscosity of air. Thus, the initially dispersed particles will experience a net downward drift due to gravity, and they will eventually reach a thermal equilibrium distribution.

(a) Find a condition on the mass m such that the equilibrium concentration of particles at the bottom of the room is twice that at the top.

(b) For particles *much heavier* than the mass found in (a), for what molecular mass m does the average height (initially $h/2$) reach a specified value αh ($\alpha < 1/2$) in a time τ?

(c) For particles with mass density of 1000 kg/m^3 (typical for organic materials), find the largest particle mass that has average height greater than 1 meter in a 3-meter-high-room after 1000 s. The viscosity of air is 10^{-5}J·s·m^{-3}.

2. Consider a gas of N atoms in volume V, each with spin-1/2. The magnitude of the magnetic moment of each atomic spin is μ. You may consider the ions to undergo classical motion and you may ignore their interactions with one another (apart from assuming that there are sufficient interactions that the spin and translational degrees of freedom can reach thermal equilibrium).

(a) Compute the partition function as a function of N, T, V, μ and external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(b) Find the total magnetic moment of the gas as a function of N, T, V, μ and B.

(c) Find the entropy of the gas as a function of N, T, V, μ and B.

(d) Suppose the gas is equilibrated at temperature T_0 in zero magnetic field, and then is thermally isolated (prohibited from exchanging energy with the exterior of the volume V). Now the magnetic field intensity is slowly increased from 0. Find the temperature of the gas $T(B)$ in the limit of small B.

3. N spin-1/2 fermions of mass m are placed in a two-dimensional harmonic potential,

$$V(x, y) = \frac{m\omega^2}{2}(x^2 + y^2)$$

at zero temperature. You may consider $N >> 1$.

(a) What is the Fermi energy of this system of particles?

(b) What is the ground state energy of the system?

(c) Estimate the radius of the "droplet" of particles in the potential, assuming that the Fermi energy is much larger than $\hbar\omega$.

(d) Suppose $N >> 1$ spin-1 bosons are placed in the same potential. Show (analytically) whether Bose condensation does or does not occur at low but nonzero temperature.

You may find the fermion and boson average occupation numbers useful:

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} \pm 1}$$

where the plus sign is for fermions and the minus sign is for bosons.

Constants

$k_B = 1.381 \times 10^{-23}$ J/K $\quad h = 6.626 \times 10^{-34}$ J·sec $\quad c = 2.998 \times 10^{8}$ m/sec

$m_p = 1.673 \times 10^{-27}$ kg $\quad m_n = 1.675 \times 10^{-27}$ kg $\quad m_e = 9.109 \times 10^{-31}$ kg

$e = 1.602 \times 10^{-19}$ C $\quad G = 6.67 \times 10^{-11}$ N m^2/kg^2

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

$$\int_{0}^{\infty} dx \ x^n \ e^{-x} = n!$$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Stirling's approximation for log of factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \mathcal{O}(1)) \qquad \ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$