Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, June 9, 2010, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.

1. Consider two infinite planes placed parallel to the xy-plane. They are thin and their separation is negligible. The lower one has a fixed charge density $\sigma > 0$. The upper one is conducting and isolated and carries no net charge. A physical dipole consisting of charges $+Q$ and $-Q$ separated a distance d is placed at infinity on the z-axis, with the positive charge a larger distance from the xy-plane than the negative charge.

   (a) What is the dipole moment of the two charges $+Q$ and $-Q$?

   Bring the dipole in along the z-axis (without rotating it) until the center of the dipole is a distance b above the two planes.

   (b) What is the work done?

   (c) What is the net force acting on the dipole?

   (d) For which values of b and σ does that force vanish?

   Peel off the plane with the surface charge density σ.

   (e) What is the dipole moment of the original dipole and the conducting plane?

   (f) With which power of r does the scalar potential Φ fall off, for $r \gg b,\ d$? (It is permissible to consider a special case and to generalize.)

2. A perfectly conducting, infinite wire carries steady current I_0 for all time.

   (a) Derive the magnetic and electric fields.

   Now consider the case in which $I(t) = 0$ for $t < 0$, and $I(t) = I_0$ for $t \geq 0$.

   (b) Derive $\vec{A}$ using retarded potentials (see formulae below).

   (c) From $\vec{A}$, calculate $\vec{B}$ and $\vec{E}$.

   (d) Show that as $t \to \infty$, your answers in (c) correspond to your answers (a).

Note: the retarded solution for the Helmholtz wave equation can be written

$$\begin{aligned}\Phi(\vec{x}, t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{[\rho(\vec{x}', t')]_{\rm ret}}{R} \\ \vec{A}(\vec{x}, t) &= \frac{\mu_0}{4\pi}\int d^3x' \, \frac{[\vec{J}(\vec{x}', t')]_{\rm ret}}{R}\end{aligned}$$

where $\vec{R} \equiv \vec{x} - \vec{x}'$ and the subscript "ret" refers to the retarded time, $t' = t - R/c$.

You may benefit from consulting the table of integrals provided for this exam.

3. An infinite plane wave propagates with $\vec{k} = k\hat{z}$ and $k = \omega n/c$ where $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$. The electric field is proportional to $E_0(\hat{x} + i\hat{y})/\sqrt{2}$.

   (a) What is $\vec{B}(\vec{x}, t)$, explicitly?

   (b) What is the time-averaged power, P, transmitted in the direction of propagation? (Hint: $\vec{S} = \vec{E} \times \vec{H}^*$ and $P = (1/2)\, Re\,(\vec{S} \cdot \hat{n})$, where $\hat{n}$ is a normal to the surface.)

   The frequency ω is adjusted to correspond to a region of anomalous dispersion, in which case $\epsilon(\omega)$, and hence $n(\omega)$, are complex.

   (c) Writing $k = k_R + ik_I$, describe what happens to the wave as it propagates along $\hat{z}$.

   (d) What is the ratio of power, $P(\text{anom.disp.})/P(\text{no disp.})$, after a propagation through 4λ, where P is the time-averaged power as in part (b) and λ is the wavelength?

   (e) Compute the ratio v_g/v_{ph}, where v_g is the group velocity and v_{ph} is the phase velocity. Normally, $v_g \leq v_{ph}$ but in the case of anomalous dispersion, one can have $v_g \gg v_{ph}$. Explain how.

4. In inertial frame K, an infinite line of charge (linear charge density $\lambda > 0$) lies at rest along the z-axis.

   (a) What are $\vec{E}$ and $\vec{B}$?

   Now consider a different inertial frame K' which is moving at velocity $-v\hat{z}$ with respect to the first frame K.

   (b) What are the charge density λ' and the current density $\vec{J}'$ in inertial frame K'?

   (c) What are $\vec{E}'$ and $\vec{B}'$?

   (d) Calculate the following quantities in both inertial frames K and K' and comment on any differences and similarities of the values in the two inertial frames.

$$u \equiv \vec{E} \cdot \vec{B} \qquad v \equiv c^2 B^2 - E^2 \qquad w \equiv c^2 \lambda^2 - J^2$$

   (e) In a general case, with both $\vec{E}$ and $\vec{B}$ given in inertial frame K, under what conditions is it possible to boost to a different frame K' in which either $\vec{E}'$ or $\vec{B}'$ are zero? Explain your answer in a few words.

(Note: This problem uses Gaussian units, in contrast to the other problems.)