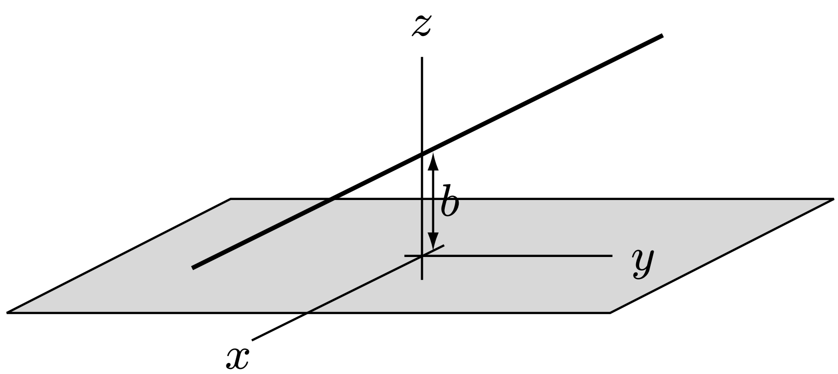
Northwestern University Physics and Astronomy Ph.D. Qualifying Examination

Wednesday, September 15, 2010, 9 am - 1 pm

Electricity and Magnetism

Solve 3 out of 4 problems
If you solve all 4 problems, only the first 3 will be graded.

Solve each problem in a separate exam solution book and write your ID number – not your name – on each book. If your solution uses more than one book, label each book with "1 out of 2", "2 out of 2" and so on.



1. A line of uniform charge density ($\lambda > 0$) runs parallel to the x-axis and passes through $(0, 0, b)$. An infinite perfectly-conducting sheet lies in the xy-plane, and is grounded.

   (a) What is the electric field $\vec{E}$ in the xy-plane? What is $\vec{E}$ on the z-axis?

   (b) Next, the charge distribution λ moves with velocity v in the $\hat{x}$-direction. What is the magnitude of the magnetic field $\vec{B}$ in the xy-plane and on the z-axis? (Neglect relativistic and time-of-propagation effects.)

   (c) Replace the line of charge by a ring of current centered on the z-axis and of radius R. The steady current is I and flows according to the right-hand rule. What is the total magnetic dipole moment?

2. A hollow non-conducting sphere of radius b is centered at the origin.

   (a) At first, $V = 0$ on the surface of the sphere, and it carries a charge Q distributed evenly on the surface. Find $V(r)$ everywhere.

   (b) Next, the charge is rearranged such that the potential on the surface varies with the polar angle:

$$V_{\text{surf}}(\cos\theta) = \frac{Q}{4\pi\epsilon_0\, b}(1 + \cos\theta).$$

   Again, find the potential $V(r, \cos\theta)$ everywhere.

   (c) What is the surface charge density, $\sigma(\cos\theta)$, given the potential in part (b)? What is the total charge?

   (d) A point charge $-q$ is placed on the z axis at $z = d$, with $d > b$. The charge on the sphere does not move. Rotate the sphere by 180° around the y-axis. What work did you do to rotate the sphere?

3. An ideal oscillating electric dipole ($\vec{p} = qd\hat{z}$) is placed at the origin, for which the retarded potentials are

$$\begin{aligned} V(r,\theta,t) &= -\frac{p\omega}{4\pi\epsilon_0\, c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega(t-\frac{r}{c})\right] \\ \vec{A}(r,\theta,t) &= -\frac{\mu_0\, p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega(t-\frac{r}{c})\right]\,\hat{z}. \end{aligned}$$

We are interested in the radiated fields for $r \gg \lambda$ (λ is the wavelength); retain only the largest terms in the calculation.

(a) Compute $\vec{E}(r,\theta,t)$ and show that it corresponds to a spherical wave with polarization in a plane that contains the $\hat{z}$-axis.

(b) Compute $\vec{B}(r,\theta,t)$ and show that $\vec{B} \perp \vec{E}$; hence, $\vec{B}$ lies in a plane parallel to the xy-plane.

(c) Compute the time-averaged radiated power, $\langle P\rangle = \int \langle\vec{S}\rangle \cdot d\vec{a}$, where $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ is the Poynting vector.

(d) Explain how your answer in (c) relates to the Larmor formula

$$P = \frac{\mu_0\, q^2\, a^2}{6\pi c}$$

where a is the acceleration. (Hint: think of two opposite charges oscillating up and down as a model for the dipole. Also, this P is not time-averaged.)

4. Consider three inertial frames, S, S' and S''. A body moves with velocity $\vec{u}$ in S. The velocity of S' with respect to S is $\vec{v}_1$, and the velocity of S'' with respect to S' is $\vec{v}_2$. In fact, $\vec{u}$, $\vec{v}_1$ and $\vec{v}_2$ are all parallel.

The formula for the relativistic addition of velocities is

$$u_1 = \frac{v_1 + u}{1 + \frac{1}{c^2} v_1 u} \tag{1}$$

giving the velocity of the body in S'. (c is the speed of light.)

(a) What is u_1 if $v_1 = c$ or $-c$?

(b) The formula for the Galilean addition of velocities is $u_1 = v_1 + u$. What is the lowest-order correction to this classical formula, assuming $v_1, u \ll c$? Does it increase or decrease u_1 with respect to the Galilean result?

(c) What is u_2 (i.e., the velocity of the body in S'') in terms of u, v_1 and v_2?

(d) Is your expression for u_2 symmetric in v_1 and v_2? Comment.

(e) Compare your answer in (c) to an incorrect answer obtained by inserting "$v_1 + v_2$" in place of "v_1" in Eq. (1). What is the correction factor to this incorrect result, to lowest order in u, v_1 and v_2, assuming these are all small compared to c and that $u \ll v_1, v_2$?

(f) Take $v_1 = c - \epsilon$ and $v_2 = -(c - \epsilon)$. What is u_2 in this case? What is the result taking $\epsilon \to 0$?

(g) Use Eq. (1) and the formula for time dilation to demonstrate that the infinitesimal four-vector is invariant:

$$(d\bar{x})^2 - c^2(d\bar{t})^2 = (dx)^2 - c^2(dt)^2.$$

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

$$\left.\begin{aligned} \bar{x}^0 &= \gamma(x^0 - \beta x^1), \\ \bar{x}^1 &= \gamma(x^1 - \beta x^0), \\ \bar{x}^2 &= x^2, \\ \bar{x}^3 &= x^3. \end{aligned}\right\}$$

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau',}$$

ne potential simplifies to

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.}$$

$P_0(x)$	=	1
$P_1(x)$	=	x
$P_2(x)$	=	$(3x^2-1)/2$
$P_3(x)$	=	$(5x^3-3x)/2$
$P_4(x)$	=	$(35x^4-30x^2+3)/8$
$P_5(x)$	=	$(63x^5-70x^3+15x)/8$

Table 3.1 Legendre Polynomials

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl : $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).}$$

the **Rodrigues formula:**

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l.$$

Maxwell's Equations

In general :

$$\begin{cases} \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{cases}$$

ϵ_0	=	$8.85\times10^{-12}\,\text{C}^2/\text{Nm}^2$	(permittivity of free space)
μ_0	=	$4\pi\times10^{-7}\,\text{N/A}^2$	(permeability of free space)
c	=	$3.00\times10^8\,\text{m/s}$	(speed of light)

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta'), \qquad (3.94)$$

where θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Substituting this back into Eq. 3.91, and noting that r is a constant, as far as the integration is concerned, I conclude that

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\,d\tau',} \qquad (3.95)$$

or, more explicitly,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right]. \qquad (3.96)$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$